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An asymptotic strain gradient Reissner-Mindlin plate model

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Abstract In this paper we derive a strain gradient plate model from the three-dimensional equations of strain gradient linearized elasticity. The deduction is based on the asymptotic analysis with respect of a small real parameter being the thickness of the elastic body we consider. The body is constituted by a second gradient isotropic linearly elastic material. The obtained model is recognized as a strain gradient Reissner-Mindlin plate model. We also provide a mathematical justification of the obtained plate model by means of a variational weak convergence result.

Keywords Asymptotic analysis · Strain gradient elasticity · Plate models · Micro-plates

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1 Introduction

Higher order gradient continuum theories in linear and nonlinear elasticity have recently raised the interest on many scientists, since modern technologies involving multi-scale materials exhibit size effects and a strong dependence on internal (material) lengths. A possible generalization of Cauchy model has been proposed in the pioneering works by Toupin, [14], Mindlin, [12], and Germain, [7]. In these papers, the stored deformation energy is assumed

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to depend not only on the strain, but also on the strain gradient. These general continua are called second gradient continua by Germain, or strain gradient continua. In second gradient continua approaches it is necessary to generalize the concept of Cauchy contact actions, see [6], and the constitutive laws, see [5]. For a general overview on second gradient elasticity theories and their applications it is worth mentioning the work by Askes and Aifantis [1].

Thin plate theories have found recently several applications in the areas of micro-mechanics and nano-mechanics. Micro-mechanical systems and nano-mechanical systems show size effects and non local behavior, hence strain gradient elasticity theories find their natural and appropriate application. Furthermore, granular materials, porous materials and, generally, materials endowed with a microstructure, in which the stresses do not depend only on the local strain, can be described by strain gradient elasticity theories. For instance, Lazopoulos has derived a mechanical model for the bending behavior of strain gradient Kirchhoff-Love plates (see [9] and [10]) and shallow shells (see [11]).

In the present paper we derive a strain gradient plate model starting from the three-dimensional equations of strain gradient linearized elasticity through an asymptotic analysis. We consider a plate-like domain filled by an isotropic second gradient linearly elastic material. By defining a small real parameter ε, which represents the thickness of the plate-like domain, we apply the asymptotic expansion method, following the approach by Ciarlet in [4]. Then, we characterize the leading terms of the asymptotic expansion and the associated limit problems. In order to have a mathematical justification of the obtained model we study the weak convergence of the solution of the three-dimensional problem towards the solution of the limit problem in a precise functional framework.

The asymptotic analysis is a widely used technique for the formal derivation and justification of classical theories of thin structures, starting from the classical three-dimensional elasticity, (see [4], in the case of plate models). For what concerns with the derivation of plate models, it is well-known that if we apply the asymptotic methods to the classical linear or nonlinear elasticity equations, we are capable to derive only Kirchhoff-Love plate models. In order to obtain the Reissner-Mindlin plate model through an asymptotic analysis or variational convergence, we need to generalize the stored elastic energy by adding some appropriate second gradient terms, see [13], or by using a different continuum model as starting point, like the micropolar continuum, see [2]. As we already mentioned, in the present approach, we use a second gradient continuum constituting the plate-like body and, by performing an asymptotic analysis, we derive a second gradient Reissner-Mindlin plate model.

The layout of this paper is as follows. In Section 2, we introduce the mathematical problem associated with the equilibrium of a strain gradient linearly elastic plate; we define a small real parameter ε which is related to the thickness of the plate; then we apply the asymptotic methods to obtain the simplified models. In Section 3, we present the main Ansatz (25) for the asymptotic expansions of the displacement field. Then we derive the limit displacement field, which corresponds to the Reissner-Mindlin kinematics, and its associated limit problem. In Section 4, we give a mathematical justification of the obtained model by presenting a weak convergence result.

2 Statement of the problem

In the sequel, Latin indices range in the set $\{1,2,3\}$, while Greek indices range in the set $\{1,2\}$ and the Einstein's summation convention with respect to the repeated indices is adopted.

Let $\omega \in^2$ be a smooth domain in the plane spanned by vectors $\mathbf{e}_\alpha$, let γ_0 be a measurable subset of the boundary γ of the set ω, such that length $\gamma_0 > 0$, and let $0 < \varepsilon < 1$ be an dimensionless *small* real parameter which will tend to zero. For each ε, we define

$$\begin{aligned}
&\Omega^\varepsilon := \omega \times (-\varepsilon, \varepsilon), \\
&\Gamma_0^\varepsilon := \gamma_0 \times [-\varepsilon, \varepsilon], \;\; \Gamma_\pm^\varepsilon := \omega \times \{\pm\varepsilon\}.
\end{aligned} \tag{1}$$

Hence the boundary of the set Ω^ε is partitioned into the lateral face $\gamma \times [-\varepsilon, \varepsilon]$ and the upper and lower faces Γ_+^ε and Γ_-^ε, and the lateral face is itself partitioned as $\gamma \times [-\varepsilon, \varepsilon] = (\gamma_0 \times [-\varepsilon, \varepsilon]) \cup (\gamma_1 \times [-\varepsilon, \varepsilon])$, where $\gamma_1 := \gamma - \gamma_0$. In order to avoid inessential complications in the sequel we suppose that $\gamma = \gamma_0$, and thus $\gamma_1 = \emptyset$.

We assume that the set $\overline{\Omega}^\varepsilon$ is the reference configuration of a strain gradient linearly elastic plate of thickness 2ε and middle surface $\overline{\omega}$. We study the physical problem corresponding to the mechanical behaviour of a strain gradient plate. The plate is completely clamped on $\Gamma_0^\varepsilon = \Gamma^\varepsilon$, in the sense that the boundary conditions of place, imposed to the displacements, are

$$u_i^\varepsilon = 0 \;\text{ and }\; \partial_n^\varepsilon u_i^\varepsilon = 0 \;\text{ on }\; \Gamma_0^\varepsilon, \tag{2}$$

where ∂_n^ε represents the derivative operator with respect to the unit outer normal vector (n_i^ε) along the boundary Γ_0^ε. Moreover, we supposed that the plate is subjected to body forces $(f_i^\varepsilon) : \Omega^\varepsilon \to \mathbb{R}^3$, and surface forces $(g_i^\varepsilon) : \Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon \to \mathbb{R}^3$.

We finally assume that the strain gradient linearly elastic material constituting the plate Ω^ε is homogeneous and isotropic. The constitutive laws for this kind of material (see [5]) take the following form:

$$\begin{aligned}
&\boldsymbol{\sigma}^\varepsilon(\mathbf{u}^\varepsilon) := \mathbb{A}^\varepsilon \mathbf{e}^\varepsilon(\mathbf{u}^\varepsilon), \\
&\mathbf{P}^\varepsilon(\mathbf{u}^\varepsilon) := \mathbb{B}^\varepsilon \nabla^\varepsilon \mathbf{e}^\varepsilon(\mathbf{u}^\varepsilon),
\end{aligned} \tag{3}$$

or, componentwise,

$$\begin{aligned}
&\sigma_{ij}^\varepsilon(\mathbf{u}^\varepsilon) = A_{ijk\ell}^\varepsilon e_{k\ell}^\varepsilon(\mathbf{u}^\varepsilon), \;\; \text{with} \\
&A_{ijk\ell}^\varepsilon := \lambda^\varepsilon \delta_{ij}\delta_{k\ell} + 2\mu^\varepsilon(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}), \\
&\\
&p_{ijk}^\varepsilon(\mathbf{u}^\varepsilon) = B_{ijk\ell pq}^\varepsilon e_{\ell pq}^\varepsilon(\mathbf{u}^\varepsilon), \;\; \text{with} \\
&B_{ijk\ell pq}^\varepsilon := c_1^\varepsilon(\delta_{ij}\delta_{k\ell}\delta_{pq} + \delta_{ij}\delta_{kp}\delta_{\ell q} + \delta_{ik}\delta_{jq}\delta_{\ell p} + \delta_{iq}\delta_{jk}\delta_{\ell p}) + c_2^\varepsilon \delta_{ij}\delta_{kq}\delta_{\ell p} + \\
&\qquad\quad + c_3^\varepsilon(\delta_{ik}\delta_{j\ell}\delta_{pq} + \delta_{ik}\delta_{jp}\delta_{\ell q} + \delta_{i\ell}\delta_{jk}\delta_{pq} + \delta_{ip}\delta_{jk}\delta_{\ell q}) + c_4^\varepsilon(\delta_{i\ell}\delta_{jp}\delta_{kq} + \\
&\qquad\quad + \delta_{ip}\delta_{j\ell}\delta_{kq}) + c_5^\varepsilon(\delta_{i\ell}\delta_{jq}\delta_{kp} + \delta_{ip}\delta_{jq}\delta_{k\ell} + \delta_{iq}\delta_{j\ell}\delta_{kp} + \delta_{iq}\delta_{jp}\delta_{k\ell}),
\end{aligned} \tag{4}$$

where $\boldsymbol{\sigma}^\varepsilon = (\sigma_{ij}^\varepsilon)$ is the classical Cauchy stress tensor, $\mathbf{P}^\varepsilon = (p_{ijk}^\varepsilon)$ is the hyperstress tensor, $\mathbf{e}^\varepsilon(\mathbf{u}^\varepsilon) = (e_{ij}^\varepsilon(\mathbf{u}^\varepsilon)) := \left(\frac{1}{2}(\partial_i^\varepsilon u_j^\varepsilon + \partial_j^\varepsilon u_i^\varepsilon)\right)$ is the linearized strain tensor and $\nabla^\varepsilon \mathbf{e}^\varepsilon(\mathbf{u}^\varepsilon) = (e_{ijk}^\varepsilon(\mathbf{u}^\varepsilon)) := (\partial_k^\varepsilon e_{ij}^\varepsilon(\mathbf{u}^\varepsilon))$ is the gradient of $\mathbf{e}^\varepsilon(\mathbf{u}^\varepsilon)$. $\mathbb{A}^\varepsilon = (A_{ijk\ell}^\varepsilon)$ and $\mathbb{B}^\varepsilon = (B_{ijk\ell pq}^\varepsilon)$ represent, respectively, the fourth order classical isotropic elasticity tensor and the sixth order isotropic strain gradient isotropic elasticity tensor. The components of the Cauchy stress tensor and the components of the hyperstress tensor can be also written as follows:

$$\begin{aligned}
&\sigma_{ij}^\varepsilon := \lambda^\varepsilon e_{pp}^\varepsilon \delta_{ij} + 2\mu^\varepsilon e_{ij}^\varepsilon, \\
&p_{ijk}^\varepsilon := c_1^\varepsilon(e_{ppj}^\varepsilon \delta_{ik} + 2e_{kpp}^\varepsilon \delta_{ij} + e_{ppi}^\varepsilon \delta_{jk}) + c_2^\varepsilon e_{ppk}^\varepsilon \delta_{ij} + \\
&\qquad + 2c_3^\varepsilon(e_{jpp}^\varepsilon \delta_{ik} + e_{ipp}^\varepsilon \delta_{jk}) + 2c_4^\varepsilon e_{ijk}^\varepsilon + 2c_5^\varepsilon(e_{ikj}^\varepsilon + e_{jki}^\varepsilon),
\end{aligned} \tag{5}$$

We assume that $\mathbb{A}^\varepsilon$ gives rise to a positive definite quadratic form on the vector space of symmetric matrices. As it is well known, this condition is satisfied if and only if $3\lambda^\varepsilon + 2\mu^\varepsilon > 0$ and $\mu^\varepsilon > 0$. We also assume that $\mathbb{B}^\varepsilon$ gives rise to a positive definite quadratic form on the vector space of all third order symmetric matrices with respect to the first two indices. In order to ensure this condition Dell'Isola et al. in [5] have proved the suffciency of the following inequalities:

$$\begin{aligned} &c_4^\varepsilon > 0, \quad -\frac{c_4^\varepsilon}{2} < c_5^\varepsilon < c_4^\varepsilon, \quad 5c_2^\varepsilon + 4c_4^\varepsilon > 2c_5^\varepsilon, \\ &c_3^\varepsilon > \frac{c_2^\varepsilon(3c_4^\varepsilon + c_5^\varepsilon) + 2(c_4^{\varepsilon 2} - 5c_1^{\varepsilon 2} - 6c_5^\varepsilon c_1^\varepsilon - 2c_5^{\varepsilon 2} + c_4^\varepsilon(2c_1^\varepsilon + c_5^\varepsilon))}{4c_5^\varepsilon - 10c_2^\varepsilon - 8c_4^\varepsilon}. \end{aligned} \tag{6}$$

To begin with, we introduce some notations that will be used in the sequel. We let:

$$\mathbf{a}\cdot\mathbf{b} := a_i b_i, \quad \mathbf{A}:\mathbf{B} := a_{ij}b_{ij} \quad \text{and} \quad \mathbf{C} \therefore \mathbf{D} := c_{ijk}d_{ijk}, \tag{7}$$

for, respectively, all vectors $\mathbf{a} = (a_i)$ and $\mathbf{b} = (b_i)$, for all symmetric second order matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, and for all third order matrices $\mathbf{C} = (c_{ijk})$ and $\mathbf{D} = (d_{ijk})$, symmetric with respect to the first two indices.

The displacement field $\mathbf{u}^\varepsilon = (u_i^\varepsilon)$ satisfies the following variational problem defined over the variable domain Ω^ε:

$$\int_{\Omega^\varepsilon} \{\sigma^\varepsilon(\mathbf{u}^\varepsilon) : \mathbf{e}^\varepsilon(\mathbf{v}^\varepsilon) + \mathbf{P}^\varepsilon(\mathbf{u}^\varepsilon) \therefore \nabla^\varepsilon \mathbf{e}^\varepsilon(\mathbf{v}^\varepsilon)\}\, dx^\varepsilon = l^\varepsilon(\mathbf{v}^\varepsilon), \tag{8}$$

for all $\mathbf{v}^\varepsilon \in V(\Omega^\varepsilon)$, where

$$V(\Omega^\varepsilon) := \left\{\mathbf{v}^\varepsilon = (v_i^\varepsilon) \in H^2(\Omega^\varepsilon;\mathbb{R}^3);\ \ \mathbf{v}^\varepsilon = \mathbf{0} \text{ and } \partial_n^\varepsilon \mathbf{v}^\varepsilon = \mathbf{0} \text{ on } \Gamma_0^\varepsilon\right\}, \tag{9}$$

and

$$l^\varepsilon(\mathbf{v}^\varepsilon) := \int_{\Omega^\varepsilon} \mathbf{f}^\varepsilon \cdot \mathbf{v}^\varepsilon dx^\varepsilon + \int_{\Gamma_\pm^\varepsilon} \mathbf{g}^\varepsilon \cdot \mathbf{v}^\varepsilon d\Gamma^\varepsilon. \tag{10}$$

Componentwise, we get:

$$\int_{\Omega^\varepsilon} \left\{\sigma_{ij}^\varepsilon(\mathbf{u}^\varepsilon) e_{ij}^\varepsilon(\mathbf{v}^\varepsilon) + p_{ijk}^\varepsilon(\mathbf{u}^\varepsilon) e_{ijk}^\varepsilon(\mathbf{v}^\varepsilon)\right\} dx^\varepsilon = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon dx^\varepsilon + \int_{\Gamma_\pm^\varepsilon} g_i^\varepsilon v_i^\varepsilon d\Gamma^\varepsilon, \tag{11}$$

for all $\mathbf{v}^\varepsilon \in V(\Omega^\varepsilon)$. We suppose that $f_i^\varepsilon \in L^2(\Omega^\varepsilon)$ and $g_i^\varepsilon \in L^2(\Gamma_\pm^\varepsilon)$.

Proposition 1 *From the assumption on the positive definiteness of the elasticity tensors $\mathbb{A}^\varepsilon$ and $\mathbb{B}^\varepsilon$, the variational problem (8) has a unique solution $\boldsymbol{u}^\varepsilon$ in $V(\Omega^\varepsilon)$.*

For a more general formulation of the three-dimensional strain gradient non linear elasticity theory, the reader can refer to the work by dell'Isola et al. [5].

In order to perform an asymptotic analysis, we need to transform problem (11) posed on a variable domain Ω^ε onto a problem posed on a fixed domain (independent of ε). Accordingly, we let

$$\begin{aligned} &\Omega := \omega \times (-1,1), \\ &\Gamma_0 := \gamma_0 \times [-1,1], \ \ \Gamma_\pm := \omega \times \{\pm 1\}. \end{aligned} \tag{12}$$

Hence, we define the following change of variables (see [4]):

$$\pi^\varepsilon : x := (\tilde{x}, x_3) \in \overline{\Omega} \mapsto x^\varepsilon := (\tilde{x}, \varepsilon x_3) \in \overline{\Omega}^\varepsilon, \text{ with } \tilde{x} = (x_\alpha). \tag{13}$$

By using the bijection π^ε, one has $\partial_\alpha^\varepsilon = \partial_\alpha$ and $\partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3$.

With the unknown displacement field $\mathbf{u}^\varepsilon = (u_i^\varepsilon) \in V(\Omega^\varepsilon)$, we associated the scaled displacement field $\mathbf{u}(\varepsilon) = (u_i(\varepsilon)) : \overline{\Omega} \to \mathbb{R}^3$ defined by:

$$u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon u_\alpha(\varepsilon)(x) \text{ and } u_3^\varepsilon(x^\varepsilon) = u_3(\varepsilon)(x) \text{ for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon. \tag{14}$$

We likewise associate with any test functions vector field $\mathbf{v}^\varepsilon = (v_i^\varepsilon) \in V(\Omega^\varepsilon)$, the scaled test functions vector field $\mathbf{v} = (v_i) : \overline{\Omega} \to \mathbb{R}^3$, defined by the scalings:

$$v_\alpha^\varepsilon(x^\varepsilon) = \varepsilon v_\alpha(x) \text{ and } v_3^\varepsilon(x^\varepsilon) = v_3(x) \text{ for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon. \tag{15}$$

We make the following assumptions on the data, and, thus, we require that the Lamé constants and the second gradient elastic moduli satisfy the following relations:

$$\lambda^\varepsilon = \lambda,\ \mu^\varepsilon = \mu,\ c_k^\varepsilon = c_k,\ k \in \{1,2,3,4,5\}. \tag{16}$$

Hence $A^\varepsilon_{ijk\ell} = A_{ijk\ell}$ and $B^\varepsilon_{ijk\ell pq} = B_{ijk\ell pq}$ are both independent of ε. We also ask that the applied body and surface forces take the following forms:

$$\begin{array}{l} f_\alpha^\varepsilon(x^\varepsilon) = \varepsilon f_\alpha(x) \text{ and } f_3^\varepsilon(x^\varepsilon) = f_3(x) \quad \text{ for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon, \\ g_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 g_\alpha(x) \text{ and } g_3^\varepsilon(x^\varepsilon) = \varepsilon g_3(x) \text{ for all } x^\varepsilon = \pi^\varepsilon x \in \overline{\Omega}^\varepsilon. \end{array} \tag{17}$$

The elastic constants λ, μ and c_k, and functions $f_i \in L^2(\Omega)$ and $g_i \in L^2(\Gamma_+ \cup \Gamma_-)$ are independent of ε.

Let us define, respectively, the rescaled components of the linearized strain tensor $e_{ij}(\mathbf{u}(\varepsilon))$ and of its gradient $e_{ijk}(\mathbf{u}(\varepsilon))$. According to the previous assumptions on the displacement field, one has:

$$\begin{array}{l} e^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon) = \varepsilon e_{\alpha\beta}(\mathbf{u}(\varepsilon)) = \frac{\varepsilon}{2}(\partial_\alpha u_\beta(\varepsilon) + \partial_\beta u_\alpha(\varepsilon)), \\ e^\varepsilon_{\alpha 3}(\mathbf{u}^\varepsilon) = \varepsilon e_{\alpha 3}(\mathbf{u}(\varepsilon)) = \frac{1}{2}(\partial_3 u_\alpha(\varepsilon) + \partial_\alpha u_3(\varepsilon)) \\ e^\varepsilon_{33}(\mathbf{u}^\varepsilon) = e_{33}(\mathbf{u}(\varepsilon)) = \frac{1}{\varepsilon}\partial_3 u_3(\varepsilon), \end{array} \tag{18}$$

and,

$$\begin{array}{l} e^\varepsilon_{\alpha\beta\gamma}(\mathbf{u}^\varepsilon) = \varepsilon e_{\alpha\beta\gamma}(\mathbf{u}(\varepsilon)) = \varepsilon\partial_\gamma e_{\alpha\beta}(\mathbf{u}(\varepsilon)), \\ e^\varepsilon_{\alpha\beta 3}(\mathbf{u}^\varepsilon) = e_{\alpha\beta 3}(\mathbf{u}(\varepsilon)) = \partial_3 e_{\alpha\beta}(\mathbf{u}(\varepsilon)), \\ e^\varepsilon_{\alpha 3\beta}(\mathbf{u}^\varepsilon) = e_{\alpha 3\beta}(\mathbf{u}(\varepsilon)) = \partial_\beta e_{\alpha 3}(\mathbf{u}(\varepsilon)), \\ e^\varepsilon_{\alpha 33}(\mathbf{u}^\varepsilon) = \frac{1}{\varepsilon} e_{\alpha 33}(\mathbf{u}(\varepsilon)) = \frac{1}{\varepsilon}\partial_3 e_{\alpha 3}(\mathbf{u}(\varepsilon)), \\ e^\varepsilon_{33\alpha}(\mathbf{u}^\varepsilon) = \frac{1}{\varepsilon} e_{33\alpha}(\mathbf{u}(\varepsilon)) = \frac{1}{\varepsilon}\partial_\alpha e_{33}(\mathbf{u}(\varepsilon)), \\ e^\varepsilon_{333}(\mathbf{u}^\varepsilon) = \frac{1}{\varepsilon^2} e_{333}(\mathbf{u}(\varepsilon)) = \frac{1}{\varepsilon^2}\partial_3 e_{33}(\mathbf{u}(\varepsilon)). \end{array} \tag{19}$$

By virtue of the relations above, we can compute the components of the rescaled hyperstress tensor $p_{ijk}(\mathbf{u}(\varepsilon)) := B_{ijk\ell pq} e_{\ell pq}(\mathbf{u}(\varepsilon))$ as follows

$$\begin{array}{l} p_{\alpha\beta\gamma}(\mathbf{u}(\varepsilon)) = \varepsilon p^1_{\alpha\beta\gamma}(\mathbf{u}(\varepsilon)) + \frac{1}{\varepsilon} p^{-1}_{\alpha\beta\gamma}(\mathbf{u}(\varepsilon)), \\ p_{\alpha\beta 3}(\mathbf{u}(\varepsilon)) = p^0_{\alpha\beta 3}(\mathbf{u}(\varepsilon)) + \frac{1}{\varepsilon^2} p^{-2}_{\alpha\beta 3}(\mathbf{u}(\varepsilon)), \\ p_{\alpha 3\beta}(\mathbf{u}(\varepsilon)) = p^0_{\alpha 3\beta}(\mathbf{u}(\varepsilon)) + \frac{1}{\varepsilon^2} p^{-2}_{\alpha 3\beta}(\mathbf{u}(\varepsilon)), \\ p_{\alpha 33}(\mathbf{u}(\varepsilon)) = \varepsilon p^1_{\alpha 33}(\mathbf{u}(\varepsilon)) + \frac{1}{\varepsilon} p^{-1}_{\alpha 33}(\mathbf{u}(\varepsilon)), \\ p_{33\alpha}(\mathbf{u}(\varepsilon)) = \varepsilon p^1_{33\alpha}(\mathbf{u}(\varepsilon)) + \frac{1}{\varepsilon} p^{-1}_{33\alpha}(\mathbf{u}(\varepsilon)), \\ p_{333}(\mathbf{u}(\varepsilon)) = p^0_{333}(\mathbf{u}(\varepsilon)) + \frac{1}{\varepsilon^2} p^{-2}_{333}(\mathbf{u}(\varepsilon)), \end{array} \tag{20}$$

where

$$\begin{aligned}
p^{1}_{\alpha\beta\gamma}(\mathbf{u}(\varepsilon)) :=& c_1[\delta_{\alpha\gamma}e_{\sigma\sigma\beta}(\mathbf{u}(\varepsilon))+2\delta_{\alpha\beta}e_{\gamma\sigma\sigma}(\mathbf{u}(\varepsilon))+\delta_{\beta\gamma}e_{\sigma\sigma\alpha}(\mathbf{u}(\varepsilon))]\\
&+c_2\delta_{\alpha\beta}e_{\sigma\sigma\gamma}(\mathbf{u}(\varepsilon))+2c_3[\delta_{\alpha\gamma}e_{\beta\sigma\sigma}(\mathbf{u}(\varepsilon))+\delta_{\beta\gamma}e_{\alpha\sigma\sigma}(\mathbf{u}(\varepsilon))]+\\
&+2c_4e_{\alpha\beta\gamma}(\mathbf{u}(\varepsilon))+2c_5[e_{\alpha\gamma\beta}(\mathbf{u}(\varepsilon))+e_{\beta\gamma\alpha}(\mathbf{u}(\varepsilon))],\\
p^{-1}_{\alpha\beta\gamma}(\mathbf{u}(\varepsilon)) :=& c_1[\delta_{\alpha\gamma}e_{33\beta}(\mathbf{u}(\varepsilon))+2\delta_{\alpha\beta}e_{\gamma33}(\mathbf{u}(\varepsilon))+\delta_{\beta\gamma}e_{33\alpha}(\mathbf{u}(\varepsilon))]+\\
&+c_2\delta_{\alpha\beta}e_{33\gamma}(\mathbf{u}(\varepsilon))+2c_3[\delta_{\alpha\gamma}e_{\beta33}(\mathbf{u}(\varepsilon))+\delta_{\beta\gamma}e_{\alpha33}(\mathbf{u}(\varepsilon))],\\
p^{0}_{\alpha\beta3}(\mathbf{u}(\varepsilon)) :=& 2c_1\delta_{\alpha\beta}e_{3\sigma\sigma}(\mathbf{u}(\varepsilon))+c_2\delta_{\alpha\beta}e_{\sigma\sigma3}(\mathbf{u}(\varepsilon))+2c_4e_{\alpha\beta3}(\mathbf{u}(\varepsilon))+\\
&+2c_5[e_{\alpha3\beta}(\mathbf{u}(\varepsilon))+e_{\beta3\alpha}(\mathbf{u}(\varepsilon))],\\
p^{-2}_{\alpha\beta3}(\mathbf{u}(\varepsilon)) :=& (2c_1+c_2)\delta_{\alpha\beta}e_{333}(\mathbf{u}(\varepsilon)),\\
p^{0}_{\alpha3\beta}(\mathbf{u}(\varepsilon)) :=& c_1\delta_{\alpha\beta}e_{\sigma\sigma3}(\mathbf{u}(\varepsilon))+2c_3\delta_{\alpha\beta}e_{3\sigma\sigma}(\mathbf{u}(\varepsilon))+2c_4e_{\alpha3\beta}(\mathbf{u}(\varepsilon))+\\
&+2c_5e_{\alpha\beta3}(\mathbf{u}(\varepsilon)),\\
p^{-2}_{\alpha3\beta}(\mathbf{u}(\varepsilon)) :=& (c_1+2c_3)\delta_{\alpha\beta}e_{333}(\mathbf{u}(\varepsilon)),\\
p^{1}_{\alpha33}(\mathbf{u}(\varepsilon)) :=& c_1e_{\sigma\sigma\alpha}(\mathbf{u}(\varepsilon))+2c_3e_{\alpha\sigma\sigma}(\mathbf{u}(\varepsilon)),\\
p^{-1}_{\alpha33}(\mathbf{u}(\varepsilon)) :=& (c_1+2c_5)e_{33\alpha}(\mathbf{u}(\varepsilon))+2(c_3+c_4+c_5)e_{\alpha33}(\mathbf{u}(\varepsilon)),\\
p^{1}_{33\alpha}(\mathbf{u}(\varepsilon)) :=& 2c_1e_{\alpha\sigma\sigma}(\mathbf{u}(\varepsilon))+c_2e_{\sigma\sigma\alpha}(\mathbf{u}(\varepsilon)),\\
p^{-1}_{33\alpha}(\mathbf{u}(\varepsilon)) :=& 2(c_1+2c_5)e_{\alpha33}(\mathbf{u}(\varepsilon))+(c_2+2c_4)e_{33\alpha}(\mathbf{u}(\varepsilon)),\\
p^{0}_{333}(\mathbf{u}(\varepsilon)) :=& (2c_1+c_2)e_{\sigma\sigma3}(\mathbf{u}(\varepsilon))+2(c_1+2c_3)e_{3\sigma\sigma}(\mathbf{u}(\varepsilon)),\\
p^{-2}_{333}(\mathbf{u}(\varepsilon)) :=& (4c_1+c_2+4c_3+2c_4+4c_5)e_{333}(\mathbf{u}(\varepsilon)).
\end{aligned} \tag{21}$$

We can now reformulate the problem on the fixed domain Ω. From Proposition 1 it follows that for every $\varepsilon > 0$ the rescaled displacement field $\mathbf{u}(\varepsilon) \in V(\Omega)$ is the unique solution of the following rescaled problem:

$$\frac{1}{\varepsilon^4}a_{-4}(\mathbf{u}(\varepsilon),\mathbf{v})+\frac{1}{\varepsilon^2}a_{-2}(\mathbf{u}(\varepsilon),\mathbf{v})+a_0(\mathbf{u}(\varepsilon),\mathbf{v})+\varepsilon^2a_2(\mathbf{u}(\varepsilon),\mathbf{v})=l_0(\mathbf{v})+\varepsilon^2l_2(\mathbf{v}), \tag{22}$$

for all $\mathbf{v} \in V(\Omega)$, where

$$V(\Omega) := \left\{\mathbf{v}=(v_i)\in H^2(\Omega;\mathbb{R}^3);\ \ \mathbf{v}=\mathbf{0} \text{ and } \partial_n\mathbf{v}=\mathbf{0} \text{ on } \Gamma_0\right\}. \tag{23}$$

The bilinear forms $a_{-4},\ a_{-2},\ a_0,\ a_2 : V(\Omega)\times V(\Omega)\to\mathbb{R}$ and the linear forms $l_0,\ l_2 : V(\Omega)\to\mathbb{R}$ are respectively defined as follows:

$$
\begin{aligned}
&a_{-4}(\mathbf{u}(\varepsilon),\mathbf{v}) := \int_\Omega p^{-2}_{333}(\mathbf{u}(\varepsilon))e_{333}(\mathbf{v})dx,\\
&a_{-2}(\mathbf{u}(\varepsilon),\mathbf{v}) := \int_\Omega [(\lambda+2\mu)e_{33}(\mathbf{u}(\varepsilon))e_{33}(\mathbf{v})+p^{-2}_{\alpha\beta 3}(\mathbf{u}(\varepsilon))e_{\alpha\beta 3}(\mathbf{v})+\\
&\qquad\qquad +2p^{-2}_{\alpha 3\beta}(\mathbf{u}(\varepsilon))e_{\alpha 3\beta}(\mathbf{v})+2p^{-1}_{\alpha 33}(\mathbf{u}(\varepsilon))e_{\alpha 33}(\mathbf{v})+\\
&\qquad\qquad +p^{-1}_{33\alpha}(\mathbf{u}(\varepsilon))e_{33\alpha}(\mathbf{v})+p^{0}_{333}(\mathbf{u}(\varepsilon))e_{333}(\mathbf{v})]dx,\\
&a_0(\mathbf{u}(\varepsilon),\mathbf{v}) := \int_\Omega [\lambda e_{\sigma\sigma}(\mathbf{u}(\varepsilon))e_{33}(\mathbf{v})+\lambda e_{33}(\mathbf{u}(\varepsilon))e_{\sigma\sigma}(\mathbf{v})+4\mu e_{\alpha 3}(\mathbf{u}(\varepsilon))e_{\alpha 3}(\mathbf{v})+\\
&\qquad\qquad +p^{-1}_{\alpha\beta\gamma}(\mathbf{u}(\varepsilon))e_{\alpha\beta\gamma}(\mathbf{v})+p^{0}_{\alpha\beta 3}(\mathbf{u}(\varepsilon))e_{\alpha\beta 3}(\mathbf{v})+2p^{0}_{\alpha 3\beta}(\mathbf{u}(\varepsilon))e_{\alpha 3\beta}(\mathbf{v})+\\
&\qquad\qquad +2p^{1}_{\alpha 33}(\mathbf{u}(\varepsilon))e_{\alpha 33}(\mathbf{v})+p^{1}_{33\alpha}(\mathbf{u}(\varepsilon))e_{33\alpha}(\mathbf{v})]dx,\\
&a_4(\mathbf{u}(\varepsilon),\mathbf{v}) := \int_\Omega [\lambda e_{\sigma\sigma}(\mathbf{u}(\varepsilon))e_{\tau\tau}(\mathbf{v})+2\mu e_{\alpha\beta}(\mathbf{u}(\varepsilon))e_{\alpha\beta}(\mathbf{v})+\\
&\qquad\qquad +p^{1}_{\alpha\beta\gamma}(\mathbf{u}(\varepsilon))e_{\alpha\beta\gamma}(\mathbf{v})]dx,\\
&l_0(\mathbf{v}) := \int_\Omega f_3v_3dx+\int_{\Gamma_\pm} g_3v_3d\Gamma,\\
&l_2(\mathbf{v}) := \int_\Omega f_\alpha v_\alpha dx+\int_{\Gamma_\pm} g_\alpha v_\alpha d\Gamma.
\end{aligned}
\tag{24}
$$

3 Asymptotic analysis

We can now perform an asymptotic analysis of the rescaled problem (22). Since it has a polynomial structure with respect to the small parameter ε, we can look for the solution of the problem as a series of powers of ε:

$$
\mathbf{u}(\varepsilon)=\mathbf{u}^0+\varepsilon^2\mathbf{u}^2+\varepsilon^4\mathbf{u}^4+\varepsilon^6\mathbf{u}^6+\dots\ . \tag{25}
$$

By substituting (25) into the rescaled problem (22), and by identifying the terms with identical power of ε, we obtain, as customary, the following set of problems, defined for all $\mathbf{v}\in V(\Omega)$:

$$
\begin{aligned}
&\mathscr{P}_{-4}: a_{-4}(\mathbf{u}^0,\mathbf{v})=0,\\
&\mathscr{P}_{-2}: a_{-4}(\mathbf{u}^2,\mathbf{v})+a_{-2}(\mathbf{u}^0,\mathbf{v})=0,\\
&\mathscr{P}_{2}: \ \ a_{-4}(\mathbf{u}^4,\mathbf{v})+a_{-2}(\mathbf{u}^2,\mathbf{v})+a_0(\mathbf{u}^0,\mathbf{v})=l_0(\mathbf{v}),\\
&\mathscr{P}_{4}: \ \ a_{-4}(\mathbf{u}^6,\mathbf{v})+a_{-2}(\mathbf{u}^4,\mathbf{v})+a_0(\mathbf{u}^2,\mathbf{v})+a_2(\mathbf{u}^0,\mathbf{v})=l_2(\mathbf{v}),\\
&\mathscr{P}_{2j}: \ a_{-4}(\mathbf{u}^{2j+4},\mathbf{v})+a_{-2}(\mathbf{u}^{2j+2},\mathbf{v})+a_0(\mathbf{u}^{2j},\mathbf{v})+a_2(\mathbf{u}^{2j-2},\mathbf{v})=0,\ j\geq 2.
\end{aligned}
\tag{26}
$$

To proceed with the asymptotic analysis we need to solve each problem above and characterize the limit displacement field $\mathbf{u}^0$ and the associated limit problem.

We start by solving problem $\mathscr{P}_{-4}$. Let us choose test functions $\mathbf{v}=\mathbf{u}^0\in V(\Omega)$:

$$
\int_\Omega p^{-2}_{333}(\mathbf{u}^0)e_{333}(\mathbf{u}^0)dx=\int_\Omega(4c_1+c_2+4c_3+2c_4+4c_5)(e_{333}(\mathbf{u}^0))^2dx=0. \tag{27}
$$

Since $4c_1+c_2+4c_3+2c_4+4c_5>0$, by virtue of the positive definiteness of $\mathbb{B}$, we get $e_{333}(\mathbf{u}^0)=0$, which implies that

$$
u^0_3(\tilde{x},x_3)=w^0(\tilde{x})+x_3b^0_3(\tilde{x}). \tag{28}
$$

Let us consider problem $\mathscr{P}_{-2}$. Since $e_{333}(\mathbf{u}^0)=0$, we get that $p^{-2}_{\alpha 3\beta}(\mathbf{u}^0)=p^{-2}_{\alpha\beta 3}(\mathbf{u}^0)=0$, thus one has:

$$\begin{aligned}\int_\Omega [p^{-2}_{333}(\mathbf{u}^2)e_{333}(\mathbf{v})+(\lambda+2\mu)e_{33}(\mathbf{u}^0)e_{33}(\mathbf{v})+2p^{-1}_{\alpha 33}(\mathbf{u}^0)e_{\alpha 33}(\mathbf{v})+\\ +p^{-1}_{33\alpha}(\mathbf{u}^0)e_{33\alpha}(\mathbf{v})+p^0_{333}(\mathbf{u}^0)e_{333}(\mathbf{v})]dx \ \text{ for all } \mathbf{v}\in V.\end{aligned} \tag{29}$$

If we choose test functions $\mathbf{v}=\mathbf{u}^0\in V(\Omega)$, problem $\mathscr{P}_{-2}$ reads as follows:

$$\begin{aligned}\int_\Omega \Bigg[(\lambda+2\mu)(e_{33}(\mathbf{u}^0))^2+4(c_3+c_4+c_5)\left(e_{\alpha 33}(\mathbf{u}^0)+\frac{2c_5+c_1}{c_3+c_4+c_5}e_{33\alpha}(\mathbf{u}^0)\right)^2+\\ +\left(c_2+2c_4-\frac{(2c_5+c_1)^2}{c_3+c_4+c_5}\right)(e_{33\alpha}(\mathbf{u}^0))^2\Bigg]dx=0.\end{aligned} \tag{30}$$

Since the assumptions on $\mathbb{A}$ and $\mathbb{B}$ imply that the coefficients multiplying the quadratic terms are positive, we obtain that

$$e_{33}(\mathbf{u}^0)=e_{33\alpha}(\mathbf{u}^0)=e_{\alpha 33}(\mathbf{u}^0)=0. \tag{31}$$

By virtue of relations (31), the displacement field $\mathbf{u}^0$ can be updated as follows:

$$\begin{cases} u^0_\alpha(\tilde{x},x_3)=\bar{u}^0_\alpha(\tilde{x})+x_3\varphi^0_\alpha(\tilde{x}),\\ u^0_3(\tilde{x},x_3)=w^0(\tilde{x}).\end{cases} \tag{32}$$

The above displacement field corresponds to the well-known Reissner-Mindlin kinematics assumptions for a plate. Since we want to focus our attention on the flexural behavior of the plate, in the sequel we neglect the in-plane displacements $\bar{u}^0_\alpha$, which are associated with the membrane behavior of the plate. Hence,

$$\begin{cases} u^0_\alpha(\tilde{x},x_3)=x_3\varphi^0_\alpha(\tilde{x}),\\ u^0_3(\tilde{x},x_3)=w^0(\tilde{x}).\end{cases} \tag{33}$$

Finally, by substituting $v_\alpha=0$ and $v_3=v_3(\tilde{x},x_3)$ in $\mathscr{P}_{-2}$, we have

$$\int_\Omega \left(p^{-2}_{333}(\mathbf{u}^2)+p^0_{333}(\mathbf{u}^0)\right)\partial_{33}v_3dx=0 \text{ for all } v_3\in V(\Omega), \tag{34}$$

which is verified when $p^{-2}_{333}(\mathbf{u}^2)=-p^0_{333}(\mathbf{u}^0)$ and so, we obtain the following characterization for u^2_3:

$$u^2_3(\tilde{x},x_3)=a^2_3(\tilde{x})+x_3b^2_3(\tilde{x})-\frac{x_3^2}{2\tilde{c}}\left[(c_1+2c_3)\partial_{\sigma\sigma}a^0_3+(3c_1+c_2+2c_3)\partial_\sigma b^0_\sigma\right](\tilde{x}), \tag{35}$$

with $\tilde{c}:=4c_1+c_2+4c_3+2c_4+4c_5$.

Problem $\mathscr{P}_0$ reads as follows:

$$\begin{aligned}\int_\Omega \Big[&p^{-2}_{333}(\mathbf{u}^4)e_{333}(\mathbf{v})+(\lambda+2\mu)e_{33}(\mathbf{u}^2)e_{33}(\mathbf{v})+p^{-2}_{\alpha\beta 3}(\mathbf{u}^2)e_{\alpha\beta 3}(\mathbf{v})+\\ &+2p^{-2}_{\alpha 3\beta}(\mathbf{u}^2)e_{\alpha 3\beta}(\mathbf{v})+2p^{-1}_{\alpha 33}(\mathbf{u}^2)e_{\alpha 33}(\mathbf{v})+p^{-1}_{33\alpha}(\mathbf{u}^2)e_{33\alpha}(\mathbf{v})+\\ &+p^0_{333}(\mathbf{u}^2)e_{333}(\mathbf{v})+\lambda e_{\sigma\sigma}(\mathbf{u}^0)e_{33}(\mathbf{v})+4\mu e_{\alpha 3}(\mathbf{u}^0)e_{\alpha 3}(\mathbf{v})+\\ &+p^0_{\alpha\beta 3}(\mathbf{u}^0)e_{\alpha\beta 3}(\mathbf{v})+2p^0_{\alpha 3\beta}(\mathbf{u}^0)e_{\alpha 3\beta}(\mathbf{v})+\\ &+2p^1_{\alpha 33}(\mathbf{u}^0)e_{\alpha 33}(\mathbf{v})+p^1_{33\alpha}(\mathbf{u}^0)e_{33\alpha}(\mathbf{v})\Big]dx=\int_\Omega f_3v_3dx+\int_{\Gamma_\pm}g_3v_3d\Gamma,\end{aligned} \tag{36}$$

for all $\mathbf{v} \in V(\Omega)$. Let us choose test functions $\mathbf{v} \in V(\Omega)$ such that $v_\alpha(\tilde{x},x_3) = \widehat{v}_\alpha(\tilde{x}) + x_3\eta_\alpha(\tilde{x})$ and $v_3(\tilde{x},x_3) = \eta_3(\tilde{x})$, i.e., $e_{333}(\mathbf{v}) = e_{33}(\mathbf{v}) = e_{\alpha 33}(\mathbf{v}) = e_{33\alpha}(\mathbf{v}) = 0$. Hence, problem $\mathscr{P}_2$ becomes:

$$\begin{aligned} &\int_\Omega \Big[\left(p^{-2}_{\alpha\beta 3}(\mathbf{u}^2) + p^0_{\alpha\beta 3}(\mathbf{u}^0) \right) e_{\alpha\beta 3}(\mathbf{v}) + 2\left(p^{-2}_{\alpha 3\beta}(\mathbf{u}^2) + p^0_{\alpha 3\beta}(\mathbf{u}^0) \right) e_{\alpha 3\beta}(\mathbf{v}) + \\ &\quad + 4\mu e_{\alpha 3}(\mathbf{u}^0) e_{\alpha 3}(\mathbf{v}) \Big] dx = \int_\Omega f_3\eta_3 dx + \int_{\Gamma_\pm} g_3\eta_3 d\Gamma \ \text{ for all } \mathbf{v} \in V(\Omega). \end{aligned} \tag{37}$$

Let $v_\alpha = 0$, then we find the first limit problem verified by w^0 and φ^0_α:

$$\begin{aligned} &h\int_\omega \Big[(C_1\partial_{\beta\beta}w^0 + C_2\partial_\beta\varphi^0_\beta)\partial_{\alpha\alpha}\eta_3 + \left(c_4(\partial_{\alpha\beta}w^0 + \partial_\beta\varphi^0_\alpha) + c_5(\partial_\alpha\varphi^0_\beta + \partial_\beta\varphi^0_\alpha) \right)\partial_{\alpha\beta}\eta_3 + \\ &\quad + \mu(\partial_\alpha w^0 + \varphi^0_\alpha)\partial_\alpha\eta_3 \Big] d\tilde{x} = \int_\omega q\eta_3 d\tilde{x}, \end{aligned} \tag{38}$$

for all $\eta_3 \in V(\omega) := \{\eta = (\eta_i) \in H^1(\omega,\mathbb{R}^2) \times H^2(\omega);\ \eta = \mathbf{0},\ \partial_\nu\eta_3 = 0 \text{ on } \gamma_0\}$ (by virtue of the assumption $\gamma_0 = \gamma$, one has $V(\omega) = H^1_0(\omega,\mathbb{R}^2) \times H^2_0(\omega)$), where

$$q(\tilde{x}) := \int_{-1}^{1} f_3(\tilde{x},x_3)dx_3 + g_3(\tilde{x},\pm 1), \tag{39}$$

and

$$\begin{aligned} C_1 &:= c_3 - \tfrac{(c_1-2c_3)^2}{\tilde{c}}, \\ C_2 &:= c_1 + c_3 - \tfrac{(3c_1+c_2+2c_3)(c_1+2c_3)}{\tilde{c}}. \end{aligned} \tag{40}$$

If we choose $v_3 = 0$ in problem (37), we obtain the second limit problem satisfied by w^0 and φ^0_α:

$$\begin{aligned} &\int_\omega \Big[\left(C_2\partial_{\beta\beta}w^0 + C_3\partial_\beta\varphi^0_\beta \right)\partial_\alpha\eta_\alpha + \mu(\partial_\alpha w^0 + \varphi^0_\alpha)\eta_\alpha + \\ &\quad + \left(c_4\partial_\beta\varphi^0_\alpha + (c_4+2c_5)(\partial_{\alpha\beta}w^0 + \partial_\beta\varphi^0_\alpha + \partial_\alpha\varphi^0_\beta) \right)\partial_\beta\eta_\alpha \Big] d\tilde{x} = 0, \end{aligned} \tag{41}$$

for all $\eta_\alpha \in V(\omega)$, where

$$C_3 := 2c_1 + c_2 + c_3 - \tfrac{(3c_1+c_2+2c_3)^2}{\tilde{c}}. \tag{42}$$

By integrating by parts problem (38) and (41), we obtain the following differential system:

$$\begin{cases} h(\mathscr{C}_1\Delta - \mu)\Delta w^0 + h(\mathscr{C}_2\Delta - \mu)\mathrm{div}\varphi^0 = q & \text{in } \omega, \\ (\mathscr{C}_2\Delta - \mu)\nabla w^0 + (\mathscr{C}_3\Delta - \mu)\varphi^0 + \mathscr{C}_4\nabla(\mathrm{div}\varphi^0) = \mathbf{0} & \text{in } \omega, \\ w^0 = 0,\ \ \partial_n w^0 = 0,\ \ \varphi^0 = \mathbf{0}, & \text{on } \gamma_0, \end{cases} \tag{43}$$

where $\Delta\phi := \partial_{\alpha\alpha}\phi$ is the two-dimensional Laplacian operator applied to ϕ, $\mathrm{div}\phi := \partial_\alpha\phi_\alpha$ is the divergence operator applied to $\phi = (\phi_\alpha)$, $\nabla\phi := (\partial_\alpha\phi)$ is the two-dimensional gradient operator applied to ϕ, and

$$\begin{aligned} &\mathscr{C}_1 := c_4 + C_1, \quad \mathscr{C}_2 := c_4 + 2c_5 + C_2, \\ &\mathscr{C}_3 := 2(c_4 + c_5), \quad \mathscr{C}_4 := c_4 + 2c_5 + C_3. \end{aligned} \tag{44}$$

Remark 1. We notice that the partial differential operator associated with system (43) is self adjoint, because it comes from a symmetric bilinear form associated with the variational problem (37).

Remark 2. In several works (see, for instance, [3]), Aifantis has proposed a simplified strain gradient isotropic linearly elastic constitutive law, in which $c_1 = c_3 = c_5 = 0$, $c_2 = \ell^2\lambda$ and $c_4 = \ell^2\mu$, where ℓ is an internal length connected to the micro-structure. The simplified strain gradient constitutive law gets the following expression:

$$\begin{aligned} &\sigma^{\varepsilon}_{ij}(\mathbf{u}^{\varepsilon}) := \lambda^{\varepsilon} e^{\varepsilon}_{pp}(\mathbf{u}^{\varepsilon})\delta_{ij} + 2\mu^{\varepsilon} e^{\varepsilon}_{ij}(\mathbf{u}^{\varepsilon}),\\ &p^{\varepsilon}_{ijk}(\mathbf{u}^{\varepsilon}) := \ell^2\partial^{\varepsilon}_k \sigma^{\varepsilon}_{ij}(\mathbf{u}^{\varepsilon}) = \ell^2\partial^{\varepsilon}_k(\lambda^{\varepsilon} e^{\varepsilon}_{pp}(\mathbf{u}^{\varepsilon})\delta_{ij} + 2\mu^{\varepsilon} e^{\varepsilon}_{ij}(\mathbf{u}^{\varepsilon})). \end{aligned} \tag{45}$$

In this particular case the limit problem takes the following form:

$$\begin{cases} h\mu(\ell^2\Delta - 1)(\Delta w^0 + \operatorname{div}\varphi^0) = q & \text{in } \omega,\\ \mu(\ell^2\Delta - 1)(\nabla w^0 + \varphi^0) + \mu\ell^2\Delta\varphi^0 + \ell^2\frac{\mu(2\mu+3\lambda)}{\lambda+2\mu}\nabla\operatorname{div}\varphi^0 = \mathbf{0} & \text{in } \omega, \end{cases} \tag{46}$$

or, analogously,

$$\begin{cases} h\mu(\ell^2\Delta - 1)(\Delta w^0 + \operatorname{div}\varphi^0) = q & \text{in } \omega,\\ \frac{12\ell^2}{h^2} D\Delta\operatorname{div}\varphi^0 = -q & \text{in } \omega, \end{cases} \tag{47}$$

where $D := \frac{\mu(\lambda+\mu)h^3}{3(\lambda+2\mu)} = \frac{Eh^3}{12(1-\nu^2)}$ is the classical rigidity modulus of the plate.

The coefficient $12(\ell/h)^2$ usually appears in strain gradient plate theories and it represents the ratio between the intrinsic length of the microstructure and the actual thickness of the plate. Its influence is high for small thicknesses, when the intrinsic length ℓ is comparable to the thickness of the plate. Besides, it has been shown in [10] that, by comparing the deflections of a classical Kirchhoff-Love plate and the deflections of a strain gradient Kirchhoff-Love plate, $12(\ell/h)^2$ has the effect of increasing the global stiffness of the plate.

4 A weak convergence results

In this section we establish a convergence result of the solution of the three-dimensional problem towards the solution of the simplified limit problem.

With the scaled displacement field $\mathbf{u}(\varepsilon) \in H^2(\Omega;\mathbb{R}^3)$, we associate the following tensors $\kappa(\varepsilon) = (\kappa_{ij}(\varepsilon))$ and $\nabla\kappa(\varepsilon) = (\kappa_{ijk}(\varepsilon)) := (\partial_k\kappa_{ij}(\varepsilon))$, with $\kappa_{ij}(\varepsilon) \in H^1(\Omega)$ and $\kappa_{ijk}(\varepsilon) \in L^2(\Omega)$, defined by

$$\begin{aligned} &\kappa_{\alpha\beta}(\varepsilon) := \varepsilon e_{\alpha\beta}(\varepsilon),\ \ \kappa_{\alpha 3}(\varepsilon) := e_{\alpha 3}(\varepsilon),\ \ \kappa_{33}(\varepsilon) := \tfrac{1}{\varepsilon} e_{33}(\varepsilon),\\ &\kappa_{\alpha\beta\gamma}(\varepsilon) := \varepsilon e_{\alpha\beta\gamma}(\varepsilon),\ \ \kappa_{\alpha\beta 3}(\varepsilon) := e_{\alpha\beta 3}(\varepsilon),\ \ \kappa_{\alpha 3\beta}(\varepsilon) := e_{\alpha 3\beta}(\varepsilon),\\ &\kappa_{\alpha 33}(\varepsilon) := \tfrac{1}{\varepsilon} e_{\alpha 33}(\varepsilon),\ \ \kappa_{33\alpha}(\varepsilon) := \tfrac{1}{\varepsilon} e_{33\alpha}(\varepsilon),\ \ \kappa_{333}(\varepsilon) := \tfrac{1}{\varepsilon^2} e_{333}(\varepsilon). \end{aligned} \tag{48}$$

With an arbitrary vector field $\mathbf{v} \in H^2(\Omega;\mathbb{R}^3)$, we likewise associate the tensors $\kappa(\varepsilon;\mathbf{v}) = (\kappa_{ij}(\varepsilon);\mathbf{v})$ and $\nabla\kappa(\varepsilon;\mathbf{v}) = (\kappa_{ijk}(\varepsilon;\mathbf{v})) := (\partial_k\kappa_{ij}(\varepsilon;\mathbf{v}))$. In particular, one has $\kappa(\varepsilon) = \kappa(\varepsilon;\mathbf{u}(\varepsilon))$ and $\nabla\kappa(\varepsilon) = \nabla\kappa(\varepsilon;\mathbf{u}(\varepsilon))$.

Then the rescaled problem (22) takes the particularly condensed form:

$$\int_{\Omega} (\mathbb{A}\kappa(\varepsilon) : \kappa(\varepsilon;\mathbf{v}) + \mathbb{B}\nabla\kappa(\varepsilon) \therefore \nabla\kappa(\varepsilon;\mathbf{v}))\,dx = l_0(v_3) + \varepsilon^2 l_2(v_\alpha), \tag{49}$$

for all $\mathbf{v} \in V(\Omega)$.

The main result of this section is claimed in the following theorem.

Theorem 1 *For each $\varepsilon > 0$, let $\mathbf{u}(\varepsilon)$ denote the (unique) solution of (49). Then*

$$\begin{array}{ll} u_3(\varepsilon) \rightharpoonup \bar{u}_3 & \text{in } H^2(\omega), \\ \partial_3 u_\alpha(\varepsilon) \rightharpoonup \bar{\varphi}_\alpha & \text{in } H^1(\omega), \end{array} \tag{50}$$

where $\bar{u}_3$ and $\bar{\varphi}_\alpha$ are the solutions of the limit problems (38)- (41)

$$\begin{aligned} &\int_\omega \left[(C_1 \partial_{\beta\beta} \bar{u}_3 + C_2 \partial_\beta \bar{\varphi}_\beta) \partial_{\alpha\alpha} \eta_3 + \left(c_4 (\partial_{\alpha\beta} \bar{u}_3 + \partial_\beta \bar{\varphi}_\alpha) + c_5 (\partial_\alpha \bar{\varphi}_\beta + \partial_\beta \bar{\varphi}_\alpha) \right) \partial_{\alpha\beta} \eta_3 + \right. \\ &\quad \left. + \mu (\partial_\alpha \bar{u}_3 + \bar{\varphi}_\alpha) \partial_\alpha \eta_3 \right] d\tilde{x} = \frac{1}{h} \int_\omega q \eta_3 d\tilde{x}, \\ &\int_\omega \left[\left(C_2 \partial_{\beta\beta} \bar{u}_3 + C_3 \partial_\beta \bar{\varphi}_\beta \right) \partial_\alpha \eta_\alpha + \mu (\partial_\alpha \bar{u}_3 + \bar{\varphi}_\alpha) \eta_\alpha + \right. \\ &\quad \left. + \left(c_4 \partial_\beta \bar{\varphi}_\alpha + (c_4 + 2c_5)(\partial_{\alpha\beta} \bar{u}_3 + \partial_\beta \bar{\varphi}_\alpha + \partial_\alpha \bar{\varphi}_\beta) \right) \partial_\beta \eta_\alpha \right] d\tilde{x} = 0, \end{aligned} \tag{51}$$

for all $\eta_i \in V(\omega) := \{\eta = (\eta_i) \in H^1(\omega, \mathbb{R}^2) \times H^2(\omega);\ \eta = \mathbf{0},\ \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\}$.

Proof For the sake of clarity the proof is divided into two parts. Let us define at first the following L^2-norms:

$$|\kappa(\varepsilon)|_{0,\Omega} := \left\{ \sum_{i,j} |\kappa_{ij}(\varepsilon)|^2_{0,\Omega} \right\}^{1/2}, \quad |\nabla\kappa(\varepsilon)|_{0,\Omega} := \left\{ \sum_{i,j,k} |\kappa_{ijk}(\varepsilon)|^2_{0,\Omega} \right\}^{1/2} \tag{52}$$

(i) By letting $\mathbf{v} = \mathbf{u}(\varepsilon)$ in (49), the variational problem takes the following simple form:

$$\int_\Omega (\mathbb{A}\kappa(\varepsilon) : \kappa(\varepsilon) + \mathbb{B}\nabla\kappa(\varepsilon) \therefore \nabla\kappa(\varepsilon))\, dx = l_0(u_3(\varepsilon)) + \varepsilon^2 l_2(u_\alpha(\varepsilon)), \tag{53}$$

By virtue of the positive definiteness of the bilinear form and, by definition of $\kappa(\varepsilon)$ and $\nabla\kappa(\varepsilon)$, one has

$$\begin{aligned} &\int_\Omega (\mathbb{A}\kappa(\varepsilon) : \kappa(\varepsilon) + \mathbb{B}\nabla\kappa(\varepsilon) \therefore \nabla\kappa(\varepsilon))\, dx \geq C \left\{ |\kappa(\varepsilon)|^2_{0,\Omega} + |\nabla\kappa(\varepsilon)|^2_{0,\Omega} \right\} \geq \\ &\geq C \Big\{ \varepsilon^2 \textstyle\sum_{\alpha,\beta} |e_{\alpha\beta}(\varepsilon)|^2_{0,\Omega} + \sum_\alpha |e_{\alpha 3}(\varepsilon)|^2_{0,\Omega} + \frac{1}{\varepsilon^2} |e_{33}(\varepsilon)|^2_{0,\Omega} + \\ &+ \varepsilon^2 \textstyle\sum_{\alpha,\beta,\gamma} |e_{\alpha\beta\gamma}(\varepsilon)|^2_{0,\Omega} + \sum_{\alpha,\beta} \left(|e_{\alpha\beta 3}(\varepsilon)|^2_{0,\Omega} + |e_{\alpha 3\beta}(\varepsilon)|^2_{0,\Omega} \right) + \\ &+ \frac{1}{\varepsilon^2} \textstyle\sum_\alpha \left(|e_{\alpha 33}(\varepsilon)|^2_{0,\Omega} + |e_{33\alpha}(\varepsilon)|^2_{0,\Omega} \right) + \frac{1}{\varepsilon^4} |e_{333}(\varepsilon)|^2_{0,\Omega} \Big\}. \end{aligned} \tag{54}$$

On the other side, by virtue of the continuity of the linear forms and since $\varepsilon \leq 1$, we get:

$$\begin{aligned} l_0(u_3(\varepsilon)) + \varepsilon^2 l_2(u_\alpha(\varepsilon)) &\leq C \left\{ \|u_3(\varepsilon)\|_{1,\Omega} + \varepsilon^2 \textstyle\sum_\alpha \|u_\alpha(\varepsilon)\|_{1,\Omega} \right\} \leq \\ &\leq C \left\{ |\kappa(\varepsilon)|^2_{0,\Omega} + |\nabla\kappa(\varepsilon)|^2_{0,\Omega} \right\}^{1/2}. \end{aligned} \tag{55}$$

In order to prove the inequality above, we notice that, since $u_i(\varepsilon) = 0$ on Γ_0, the norm $\|u_i(\varepsilon)\|_{1,\Omega}$ is equivalent to $\{\sum_j |\partial_j u_i(\varepsilon)|^2_{0,\Omega}\}^{1/2}$. Since $\partial_n u_i(\varepsilon) = 0$ on Γ_0 and $\mathbf{n} = (n_1, n_2, 0)$, one has the same inequality for any $\partial_\alpha u_i(\varepsilon)$. In particular we get

$$\begin{aligned} |\partial_\alpha u_3(\varepsilon)|^2_{0,\Omega} &\leq C\{\textstyle\sum_\beta |\partial_{\alpha\beta} u_3(\varepsilon)|^2_{0,\Omega} + |\partial_{\alpha 3} u_3(\varepsilon)|^2_{0,\Omega}\}, \\ |\partial_\alpha u_\beta(\varepsilon)|^2_{0,\Omega} &\leq C\{\textstyle\sum_\gamma |\partial_{\alpha\gamma} u_\beta(\varepsilon)|^2_{0,\Omega} + |\partial_{\alpha 3} u_\beta(\varepsilon)|^2_{0,\Omega}\}. \end{aligned} \tag{56}$$

Since $\partial_{\alpha j} u_i(\varepsilon) = e_{\alpha ij}(\varepsilon) + e_{ji\alpha}(\varepsilon) - e_{\alpha ji}(\varepsilon)$, one has

$$\begin{array}{l} \sum_\alpha |\partial_\alpha u_3(\varepsilon)|^2_{0,\Omega} \leq C\left\{\sum_{\alpha,\beta}\left(|e_{\alpha\beta 3}(\varepsilon)|^2_{0,\Omega} + |e_{\alpha 3\beta}(\varepsilon)|^2_{0,\Omega} + |e_{\alpha 33}(\varepsilon)|^2_{0,\Omega} + |e_{33\alpha}(\varepsilon)|^2_{0,\Omega}\right)\right\}, \\ \sum_\alpha |\partial_\alpha u_\beta(\varepsilon)|^2_{0,\Omega} \leq C\left\{\sum_{\alpha,\beta,\gamma}\left(|e_{\alpha\beta\gamma}(\varepsilon)|^2_{0,\Omega} + |e_{\alpha 3\beta}(\varepsilon)|^2_{0,\Omega} + |e_{\alpha\beta 3}(\varepsilon)|^2_{0,\Omega}\right)\right\}. \end{array} \tag{57}$$

Hence we have that

$$\begin{array}{l} \|u_3(\varepsilon)\|_{1,\Omega} + \varepsilon^2 \sum_\alpha \|u_\alpha(\varepsilon)\|_{1,\Omega} \leq \\ \leq C\left\{\frac{1}{\varepsilon^2}|e_{33}(\varepsilon)|^2_{0,\Omega} + \sum_{\alpha,\beta}\left(|e_{\alpha\beta 3}(\varepsilon)|^2_{0,\Omega} + |e_{\alpha 3\beta}(\varepsilon)|^2_{0,\Omega}\right) + \right. \\ + \frac{1}{\varepsilon^2}\sum_\alpha\left(|e_{\alpha 33}(\varepsilon)|^2_{0,\Omega} + |e_{33\alpha}(\varepsilon)|^2_{0,\Omega}\right) + \varepsilon^2 \sum_{\alpha,\beta}|e_{\alpha\beta}(\varepsilon)|^2_{0,\Omega} + \\ \left. + \varepsilon^2 \sum_{\alpha,\beta,\gamma}|e_{\alpha\beta\gamma}(\varepsilon)|^2_{0,\Omega}\right\}^{1/2} \leq C\left\{|\kappa(\varepsilon)|^2_{0,\Omega} + |\nabla\kappa(\varepsilon)|^2_{0,\Omega}\right\}^{1/2}. \end{array} \tag{58}$$

The inequalities above imply that the norms $|\kappa(\varepsilon)|_{0,\Omega}$ and $|\nabla\kappa(\varepsilon)|_{0,\Omega}$ are bounded independently of ε. Since the sequences $(\kappa_{ij}(\varepsilon))_{\varepsilon>0}$ and $(\kappa_{ijk}(\varepsilon))_{\varepsilon>0}$ are bounded in $L^2(\Omega)$, there exist a constant C such that

$$\begin{array}{lll} |e_{333}(\varepsilon)|_{0,\Omega} \leq C\varepsilon^2, & |e_{\alpha 33}(\varepsilon)|_{0,\Omega} \leq C\varepsilon, & |e_{\alpha\beta\gamma}(\varepsilon)|_{0,\Omega} \leq \frac{C}{\varepsilon}, \\ |e_{33\alpha}(\varepsilon)|_{0,\Omega} \leq C\varepsilon, & |e_{\alpha 3\beta}(\varepsilon)|_{0,\Omega} \leq C, & |e_{\alpha\beta 3}(\varepsilon)|_{0,\Omega} \leq C, \\ |e_{33}(\varepsilon)|_{0,\Omega} \leq C\varepsilon, & |e_{\alpha 3}(\varepsilon)|_{0,\Omega} \leq C, & |e_{\alpha\beta}(\varepsilon)|_{0,\Omega} \leq \frac{C}{\varepsilon}. \end{array} \tag{59}$$

Hence, from the first set of inequalities, we obtain that $\frac{1}{\varepsilon^2}e_{333}(\varepsilon) \rightharpoonup \bar{e}_{333}$, $e_{333}(\varepsilon) := \partial_{33}u_3(\varepsilon) \to 0$, $e_{33\alpha}(\varepsilon) := \partial_{\alpha 3}u_3(\varepsilon) \to 0$ and $e_{33}(\varepsilon) := \partial_3 u_3(\varepsilon) \to 0$ in $L^2(\Omega)$, and thus $\partial_3 u_3(\varepsilon) \to 0$ in $H^1(\Omega)$. Moreover, one has $\partial_\alpha u_3(\varepsilon) \rightharpoonup \bar{z}_\alpha(\tilde{x})$ in $L^2(\Omega)$.

¿From the second set of inequalities, we get that $e_{\alpha 3}(\varepsilon) \rightharpoonup \bar{e}_{\alpha 3}$, $e_{\alpha 33}(\varepsilon) := \partial_3 e_{\alpha 3}(\varepsilon) \to 0$ and $e_{\alpha 3\beta}(\varepsilon) := \partial_\beta e_{\alpha 3}(\varepsilon) \rightharpoonup \bar{e}_{\alpha 3\beta}$ in $L^2(\Omega)$. Thus $e_{\alpha 3}(\varepsilon) \rightharpoonup \bar{e}_{\alpha 3}(\tilde{x})$ in $H^1(\Omega)$ and so, $\bar{e}_{\alpha 3\beta} = \partial_\beta \bar{e}_{\alpha 3}(\tilde{x})$. By definition of $e_{\alpha 3}(\varepsilon)$, we obtain that $\partial_3 u_\alpha(\varepsilon) \rightharpoonup 2\bar{e}_{\alpha 3}(\tilde{x}) - \bar{z}_\alpha(\tilde{x})$ in $L^2(\Omega)$.

Thanks to (59), we notice that $\partial_{\alpha\beta}u_3(\varepsilon) = e_{\alpha 3\beta}(\varepsilon) + e_{\beta 3\alpha}(\varepsilon) - e_{\alpha\beta 3}(\varepsilon)$ is bounded in $L^2(\Omega)$. Therefore, $|\partial_{ij}u_3(\varepsilon)|_{0,\Omega} \leq C$ and by (57) one has $u_3(\varepsilon)$ bounded in $H^2_0(\omega)$ and so:

$$u_3(\varepsilon) \rightharpoonup \bar{u}_3(\tilde{x}) \text{ in } H^2_0(\omega). \tag{60}$$

The limit $\bar{u}_3 = \bar{u}_3(\tilde{x})$ is independent of x_3. This implies that $\bar{z}_\alpha = \partial_\alpha \bar{u}_3$ and, thus

$$\partial_3 u_\alpha(\varepsilon) \rightharpoonup \bar{\varphi}_\alpha(\tilde{x}) := 2\bar{e}_{\alpha 3}(\tilde{x}) - \partial_\alpha \bar{u}_3(\tilde{x}) \text{ in } H^1_0(\omega). \tag{61}$$

Finally, from the third set of inequalities, we deduce that $\varepsilon e_{\alpha\beta\gamma}(\varepsilon) \rightharpoonup \bar{e}_{\alpha\beta\gamma}$, $\varepsilon e_{\alpha\beta}(\varepsilon) \rightharpoonup \bar{e}_{\alpha\beta}$ and $e_{\alpha\beta 3}(\varepsilon) \rightharpoonup \bar{e}_{\alpha\beta 3}$ in $L^2(\Omega)$. We notice that $\bar{e}_{\alpha\beta 3} = \partial_\alpha \bar{e}_{\beta 3} + \partial_\beta \bar{e}_{\alpha 3} - \partial_{\alpha\beta}\bar{u}_3 = \frac{1}{2}(\partial_\alpha \bar{\varphi}_\beta + \partial_\beta \bar{\varphi}_\alpha)$ in $L^2(\Omega)$.

(ii) Now we characterize the limits $\bar{u}_3$ and $\bar{\varphi}_\alpha$. Let us consider the rescaled variational problem (22). Let multiply it by ε^2 and let ε tend to zero. Then we find that

$$\int_\Omega (\tilde{c}\bar{e}_{333} + (2c_1 + c_2)\bar{e}_{\sigma\sigma 3} + 2(c_1 + 2c_3)\partial_\sigma \bar{e}_{3\sigma})\, e_{333}(\mathbf{v})dx = 0, \tag{62}$$

for all $\mathbf{v} \in V(\Omega)$. This relation is satisfied when $\tilde{c}\bar{e}_{333} + (2c_1 + c_2)\bar{e}_{\sigma\sigma 3} + 2(c_1 + 2c_3)\partial_\sigma \bar{e}_{3\sigma} = 0$, which implies that

$$\begin{array}{l} \bar{e}_{333} = -\dfrac{2c_1 + c_2}{\tilde{c}}\bar{e}_{\sigma\sigma 3} - \dfrac{2(c_1 + 2c_3)}{\tilde{c}}\partial_\sigma \bar{e}_{3\sigma} = \\ \quad = -\dfrac{3c_1 + c_2 + c_3}{\tilde{c}}\partial_\sigma \bar{\varphi}_\sigma - \dfrac{c_1 + 2c_3}{\tilde{c}}\partial_{\sigma\sigma}\bar{u}_3. \end{array} \tag{63}$$

Let us choose test functions $\mathbf{v} \in V(\Omega)$, such that $v_\alpha(\tilde{x}, x_3) = x_3\eta_\alpha(\tilde{x})$ and $v_3(\tilde{x}, x_3) = v_3(\tilde{x})$, i.e. $\mathbf{v}$ has the same form of the limit displacement field $\mathbf{u}^0$. The variational problem (22) takes the following expression when ε tends to zero:

$$
\begin{aligned}
\int_\Omega & [((2c_1+c_2)\bar{e}_{333} + 2c_1\partial_\sigma\bar{e}_{\sigma 3} + c_2\bar{e}_{\sigma\sigma 3})e_{\tau\tau 3}(\mathbf{v}) + \\
& +(2c_4\bar{e}_{\alpha\beta 3} + 2c_5(\partial_\beta\bar{e}_{\alpha 3} + \partial_\alpha\bar{e}_{\beta 3}))e_{\alpha\beta 3}(\mathbf{v}) + \\
& +(2(c_1+2c_3)\bar{e}_{333} + 2c_3\partial_\sigma\bar{e}_{\sigma 3} + c_1\bar{e}_{\sigma\sigma 3})e_{\tau 3\tau}(\mathbf{v}) + \\
& +2(c_4\partial_\beta\bar{e}_{\alpha 3} + c_5\bar{e}_{\alpha\beta 3})e_{\alpha 3\beta}(\mathbf{v})\big] = l_0(v_3).
\end{aligned} \tag{64}
$$

By choosing $v_\alpha = 0$ and $v_3 = \eta_3(\tilde{x})$, by virtue of (63) and of the previous convergences, we obtain the first limit problem

$$
\begin{aligned}
\int_\omega & \big[(C_1\partial_{\beta\beta}\bar{u}_3 + C_2\partial_\beta\bar{\varphi}_\beta)\partial_{\alpha\alpha}\eta_3 + \big(c_4(\partial_{\alpha\beta}\bar{u}_3 + \partial_\beta\bar{\varphi}_\alpha) + c_5(\partial_\alpha\bar{\varphi}_\beta + \partial_\beta\bar{\varphi}_\alpha)\big)\,\partial_{\alpha\beta}\eta_3 + \\
& +\mu(\partial_\alpha\bar{u}_3 + \bar{\varphi}_\alpha)\partial_\alpha\eta_3]\,d\tilde{x} = \frac{1}{h}\int_\omega q\eta_3 d\tilde{x}.
\end{aligned} \tag{65}
$$

for all $\eta_3 \in V(\omega)$. Otherwise, by choosing $v_\alpha = x_3\eta_\alpha(\tilde{x})$ and $v_3 = 0$, by means of relation (63) and by means of the convergence results obtained in (i), we obtain the second limit problem

$$
\begin{aligned}
\int_\omega & \big[\big(C_2\partial_{\beta\beta}\bar{u}_3 + C_3\partial_\beta\bar{\varphi}_\beta\big)\,\partial_\alpha\eta_\alpha + \mu(\partial_\alpha\bar{u}_3 + \bar{\varphi}_\alpha)\eta_\alpha + \\
& + ((c_4+2c_5)(\partial_{\alpha\beta}\bar{u}_3 + \partial_\beta\bar{\varphi}_\alpha + \partial_\alpha\bar{\varphi}_\beta) + c_4\partial_\beta\bar{\varphi}_\alpha)\partial_\beta\eta_\alpha\big]\,d\tilde{x} = 0.
\end{aligned} \tag{66}
$$

for all $\eta_\alpha \in V(\omega)$. This completes the proof.

5 Concluding remarks

In the present work we derive a strain gradient Reissner-Mindlin plate model by means of an asymptotic analysis starting from the equations of strain gradient linearized elasticity. Besides, we give a formal justification of the simplified model by virtue of a weak convergence result. We concentrate our attention to the flexural behavior, by neglecting at this stage the membrane behaviour.

As it is mentioned in the Introduction to this paper, in order to obtain the Reissner-Mindlin kinematics for a plate through the asymptotic analysis or Γ-convergence, we need to generalize the initial classical elastic energy by adding some second gradient extra terms or by using the micropolar continuum model. In our work we give another proof of the fact that, in order to formally deduce the Reissner-Mindlin plate model, it is necessary to use a different continuum model, such as the second gradient continuum model.

¿From a mechanical point of view, the obtained strain gradient Reissner-Mindlin plate model can be used to study the mechanical behaviour of micro-plates and nano-plates, especially by considering the reduced Aifantis strain gradient elastic material. The new model contains a material internal length scale parameter ℓ to account the microstructural effect, unlike the classical Reissner-Mindlin plate model. The presence of this additional material constant enables this model to capture size effects.

At last let us explicitly point out that all the obtained conclusions are true also when the plate is clamped only on a measurable subset $\gamma_0 \times [-\varepsilon, \varepsilon]$ of the lateral boundary Γ^ε with length $\gamma_0 > 0$, and $\gamma_1 \neq \emptyset$.

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